Energy Balance Models

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Module Summary. This module introduces the student to the process of mathematical modeling. It shows how the process starts in the "real world" with a physical system and some observations or an experiment. When the laws of physics that are thought to govern the behavior of the system are translated in mathematical terms, the result is what is called a *mathematical model*. The mathematical model is subsequently analyzed for its properties and used to generate predictions about the behavior of the system in a changing environment. These predictions are tested against observations, and if there is agreement between predictions and observations, the model is accepted; otherwise, the model is refined, for example by bringing in more details of the physics, and the process is repeated. Thus, mathematical modeling is an *iterative process.*

To illustrate this iterative process, this module builds a series of *zero-dimensional energy balance models* for the Earth's climate system. In a zero-dimensional energy balance model, the Earth's climate system is described in terms of a single variable, namely the temperature of the Earth's surface averaged over the entire globe. In general, this variable varies with time; its time evolution is governed by the amount of energy coming in from the Sun (in the form of ultraviolet radiation) and the amount of energy leaving the Earth (in the form of infrared radiation). The mathematical challenge is to find expressions for the incoming and outgoing energy that are consistent with the observed current state of the climate system on Earth, that corresponds to the average temperature on Earth of 16 degrees Celsius.

Informal Description. This module introduces the student to the mathematical modeling process by showing how to build a zero-dimensional energy balance model for the Earth's climate system. The process is an iterative one and generates various versions of the model. Successive versions include more physics to better match the observations. The emphasis in the module is on the *process,* rather than the models derived in the process, because the process is universal and independent of the complexity of the model. The process is illustrated in Figure 1.

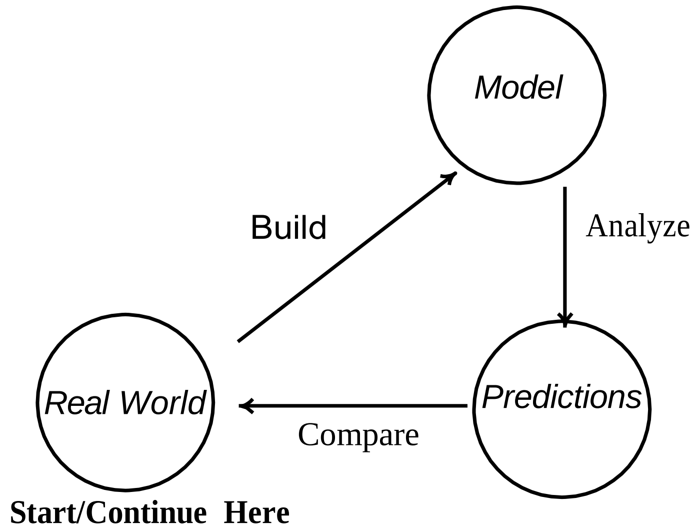


***Figure 1: The modeling cycle*.**

The mathematical modeling process starts in the "real world" with a physical system and some observations or an experiment. We assume that the behavior of the system is governed by the laws of nature—Newton's law of motion, Fourier's law of heat conduction, etc. When these laws are formulated in mathematical terms, we obtain what we call a "mathematical model"—a set of mathematical equations that describe the state of the physical system as it evolves in time. In the next step of the modeling process, we "analyze" the model—that is, we apply our mathematical knowledge to extract information from the model, to see whether we understand and can explain what we see in the real world. In the third step we use the model to make predictions about what we will see in additional experiments and observations. We then return to the real world to test these predictions by running the experiments or collecting more observations, and either accept the model if we find that the outcome matches our predictions, or refine the model if we find that improvements are needed. Typically, we go around this modeling cycle many times, building progressively better models, thus improving our understanding of the physical system and increasing our ability to make predictions about its behavior.

In this module, the physical system of interest is the Earth's climate system—a proto-typical "complex system" that has many components: the atmosphere, oceans, lakes and other bodies of water, snow and ice, land surface, all living things, and so on. The components interact and influence each other in ways that we don't always understand, so it is difficult to see how the system as a whole evolves, let alone why it evolves the way it does. For some complex system it is possible to build a physical model and observe what happens if the environment changes. This is the case, for example, for a school of fish whose behavior we can study in an aquarium. It is also true for certain aspects of human behavior, which we can study in a social network. But in climate science this is not possible; we have only one Earth, and we cannot perform a controlled real-life experiment. The best we can do if we want to gain insight into what might have happened to the Earth's climate system in the past, or what might happen to it in the future, is to build mathematical models and "play" with them. Mathematical models are the climate scientist's only experimental tools.

The modeling process—building and testing a series of imperfect models—is the most essential brick in the foundation of climate science and an indispensable tool to evaluate the arguments for or against climate change. Models are never perfect—at best, they provide some understanding and some ability to test "what-if" scenarios. Especially in an area as complex as the Earth's climate, we cannot and should not expect perfection. Recognizing and identifying imperfection and uncertainty are key parts of all modeling and, especially, climate modeling.

Mathematical models of the Earth's climate system come in many flavors. They can be simple—simple enough that we can use them for back-of-the-envelope calculations, or they can be so complicated that we need a supercomputer to learn what we want to know. But whatever kind of models we use, we should always keep in mind that they are *simplified representations* of the real world, they are not the "real world," and they are made for a purpose, namely to better understand what is driving our climate system.

The present module looks at the simplest possible description of the Earth's climate system. In the following models, the state of the climate system is characterized by a single variable—the temperature of the Earth's surface, averaged over the entire globe (referred to as "*zero-dimensional energy balance*" models in physics). An *energy balance equation* is a formal statement of the fact that the temperature of the Earth increases if the Earth receives more energy from the Sun than it re-emits into space, and that it decreases if the opposite is the case. The module shows how to construct energy balance models by finding mathematical expressions for the incoming and outgoing energy. The models are tested against "real-world" data and improved in successive steps of the iterative modeling process to better match the available data.

In this module, the focus is on the physics, but we emphasize that modeling the Earth's climate system is fundamentally an interdisciplinary activity. Understanding the Earth's climate requires knowledge, skills, and perspectives from multiple disciplines. For example, atmospheric chemistry explains why much of the incoming energy from the Sun (largely in the ultraviolet and visible regions of the spectrum) passes through the atmosphere and reaches the Earth's surface, but much of the black-body radiation emitted by the Earth (largely in the infrared regions of the spectrum) is trapped by greenhouse gases like water vapor and carbon dioxide. Similarly, the life sciences help us understand the part played by the biosphere in the Earth's cli-mate system—the effects of the biosphere on the

Earth’s albedo and the interactions between atmospheric chemistry and plant and animal life.

Target Audience. This module is suitable Lab for undergraduate students in an introductory differential equation class.

Prerequisites. Basic knowledge of the concept of derivatives and ordinary differential equations.

Mathematical Fields. Ordinary differential equations.

Applications Areas. Geophysics and climate science.

Goals and Objectives.

- Teach the process of “mathematical modeling.
- Show how a simple model like a single variable energy balance model can provide insight into aspects of climate dynamics.

1 Climate Model – Cycle #1

We consider the Earth with its atmosphere, oceans, and all other components of the climate system as a homogeneous solid sphere, ignoring differences in the atmosphere's composition (clouds!), differences among land and oceans, differences in topography (altitude), and many other things.

1.1 Observation

The climate system is powered by the Sun, which emits radiation in the ultraviolet (UV) regime (wavelength less than $0.4\,\mu$m). This energy reaches the Earth's surface, where it is converted by physical, chemical, and biological processes to radiation in the infrared (IR) regime (wavelength greater than $5\,\mu$m). This IR radiation is then reemitted into space. If the Earth's climate is in equilibrium (steady state), the average temperature of the Earth's surface does not change, so the amount of energy received must equal the amount of energy re-emitted.

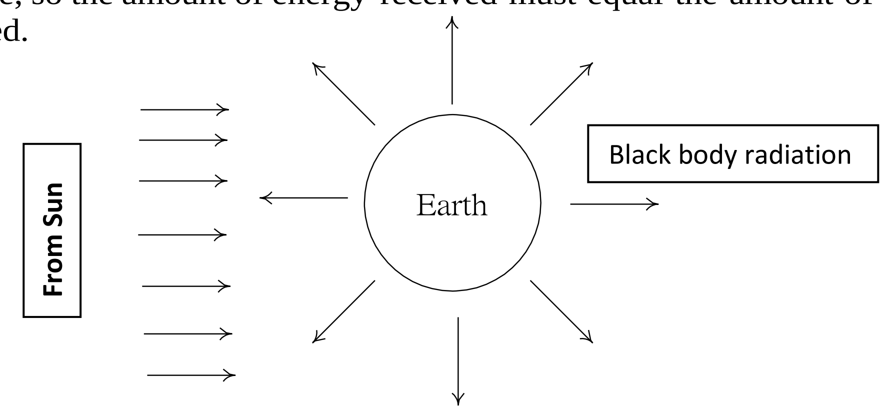


Figure 2. Simplest Climate Model

1.2 Modeling

Units:

meter (m) for length, watt (W) for energy; and kelvin (K) for temperature. Water freezes at 273.15 K which is equivalent to 0 degrees Celsius, with the increase of one degree Celsius being the same as an increase of one degree Kelvin;

Variables:

- T, the temperature of the Earth's surface averaged over the entire globe.

Building the model.

- Viewed from the Sun, the Earth is a disk (Figure 2).
- The area of the disk as seen by the Sun is πR^2, where R is the radius of the Earth.
- S, the energy flux density (also referred to as the energy flux) amount of energy (W) flowing through a flat surface of area $1\,m^2$. From satellite observations we know that the energy flux from the Sun is $S = 1367.6\,Wm^{-2}$. It is customary to define $Q = \frac{1}{4} S$ and use Q instead of S .

- The amount of energy flowing through the disk (i.e., reaching the Earth) is Incoming energy (W): $E_{in} = \pi R^2 S = 4\pi R^2 Q$ All bodies radiate energy in the form of electromagnetic radiation.
- The amount of energy radiated out depends on the temperature of the body.
- In physics, it is shown that for "black-body radiation" the temperature dependence is given by the Stefan–Boltzmann law (in units of Wm^{-2}),

$$F_{SB}(T) = \sigma T^4 \qquad (1)$$

(The subscript SB refers to the mathematical physicists Joseph Stefan and Ludwig Boltzmann, who first proposed this formula in the 1880s.)

σ (Greek, pronounced "sigma"), Stefan–Boltzmann constant; its value is $\sigma = 5.67 \cdot 10^{-8}\, Wm^{-2}K^{-4}$.

- The area of the Earth's surface is $4\pi R^2$.
- The amount of energy radiated out by the Earth is

Outgoing energy (W): $E_{out} = 4\pi R^2 \sigma T^4$.

Conclusion

$$\boldsymbol{E_{in} = 4\pi R^2 Q}$$
$$\boldsymbol{E_{out} = 4\pi R^2 \sigma T^4}$$

1.3 Analysis

If the incoming energy is greater than the outgoing energy, the Earth's temperature increases. If the incoming energy is lower than the outgoing energy, the Earth's temperature decreases. If the incoming energy balances the outgoing energy, the Earth's temperature remains constant; the planet is said to be in *thermal equilibrium.* At thermal equilibrium, the temperature T must be such that $E_{in} = E_{out}$.

Our mathematical model gives the equation

$$4\pi R^2 Q = 4\pi R^2 \sigma T^4$$

$$Q = \sigma T^4.$$

Solving for T, we obtain the expression

$$T = \left(\frac{Q}{\sigma}\right)^{\frac{1}{4}}$$

With $\sigma = 5.67 \cdot 10^{-8}$ and $S = 1376.6$, we find $T \approx 278.7$

Conclusion. Model #1 gives the average temperature at equilibrium $T \approx 278.7\,K$, about 5.5 degrees Celsius.

2 Climate Model – Cycle #2

The value $T \approx 5.5$ degrees Celsius seems reasonable but is not in agreement with the known average temperature of the Earth, which is about 16 degrees Celsius. We need a better model.

2.1 Observation

Model #1 omitted a number of important factors. The fi factor we want to add involves *reflection*—some of the incoming energy from the Sun is reflected back out into space. Snow, ice, and clouds, for example, reflect a great deal of the incoming light from the Sun. We use the term *albedo* to measure the Earth's reflectivity.

2.2 Modeling

Additional physical constants.

- α, albedo. The Earth's average albedo is about 0.3, which means that roughly 70% of the incoming energy is absorbed by the Earth's surface.

Building the model.

- The amount of energy reaching the Earth is
Incoming energy (W): $E_{in} = 4\pi R^2 Q(1 - \alpha)$.
- The amount of energy radiated out by the Earth is
Outgoing energy (W): $E_{out} = 4\pi R^2 \sigma T^4$.

2.3 Analysis

At thermal equilibrium, the temperature must be such that $E_{in} = E_{out}$. Our mathematical model gives the equation

$$Q(1 - \alpha) = \sigma T^4.$$

Exercise: Solve for T.

With $\alpha = 0.3$

$$T \approx 254.9\,K.$$

Conclusion. Although Model #2 is better, in the sense that it includes more physics, its prediction of the temperature value at equilibrium is worse than the prediction of Model #1.

3 Climate Model – Cycle #3

It is somewhat disconcerting that we construct a better model and get a result that is not as good as that of the earlier model. But once we accept the mathematical model, we must accept the result. The only option is to look where we might have overlooked something in the model. In this cycle, we focus on the outgoing radiation.

3.1 Observation

Greenhouse gases like carbon dioxide, methane, and water, as well as dust and aerosols have a significant effect on the properties of the atmosphere. The effect on the outgoing radiation is difficult to model, but the simplest approach is to reduce the Stefan–Boltzmann law by some factor.

3.2 Modeling

Additional physical parameter.

- ε, greenhouse factor ($0 < \varepsilon < 1$). This artificial *parameter* has no immediate physical meaning. It is introduced to model the effect of greenhouse gases on the permittivity of the atmosphere; its value is unknown.

Building the model.

- The amount of energy reaching the Earth is
Incoming energy (W): $E_{in} = 4\pi R^2 Q(1 - \alpha)$.
- The amount of energy radiated out by the Earth is
Outgoing energy (W): $E_{out} = 4\pi R^2 \varepsilon\sigma T^4$.

3.3 Analysis

At thermal equilibrium, the temperature must be such that $E_{\text{in}} = E_{\text{out}}$. Our mathematical model gives the equation:

$$Q(1 - \alpha) = \varepsilon\sigma T^4. \tag{2}$$

Exercise: Solve for T.

Question. Which value of ε gives a climate model that correctly predicts the current global average temperature $T^* \approx 288\ K$? (Take $\alpha_1 = 0.3$ as before).
[Answer: $\varepsilon = 0.66$]

Question. What happens if the combined effects of greenhouse gases, dust, and aerosols reduce the parameter ε from 0.66 to 0.5?

[Answer: The equilibrium temperature T increases.]

Our climate model predicts that, if the amount of greenhouse gasses in the Earth's atmosphere increases, then the Earth will warm up. This is the well-known *greenhouse gas effect.* However, this model is certainly too simple to predict the state of our planet with any great accuracy, so we should interpret this finding with great care.
An interesting question is what actually happens when the balance of incoming and outgoing energy is perturbed. Perhaps a volcanic eruption throws dust into the atmosphere, or humans release increasing amounts of CO_2 or other greenhouse gases into the atmosphere. Greenhouse gases affect the Earth's climate by absorbing some of the outgoing radiation.

Question. What do you expect to happen to the Earth's temperature if $E_{\text{in}} > E_{\text{out}}$? What if $E_{\text{out}} > E_{\text{in}}$?
[Answer: The temperature increases if $E_{\text{in}} > E_{\text{out}}$, decreases if $E_{\text{out}} > E_{\text{in}}$]
We can ask more questions. Will the temperature continue to increase or will it eventually level off at a higher value? What does the diff $E_{\text{in}} - E_{\text{out}}$ represent? How fast will the temperature change?
To answer these questions, we need a fancier model.

4. Modeling the dynamics

The simplest model assumes that the temperature changes at a rate proportional to the energy imbalance.

Question. Rewrite the last sentence as a mathematical equation.
[Answer: (most likely) $\frac{dT}{dt} = k\,(E_{in} - E_{out})$.]

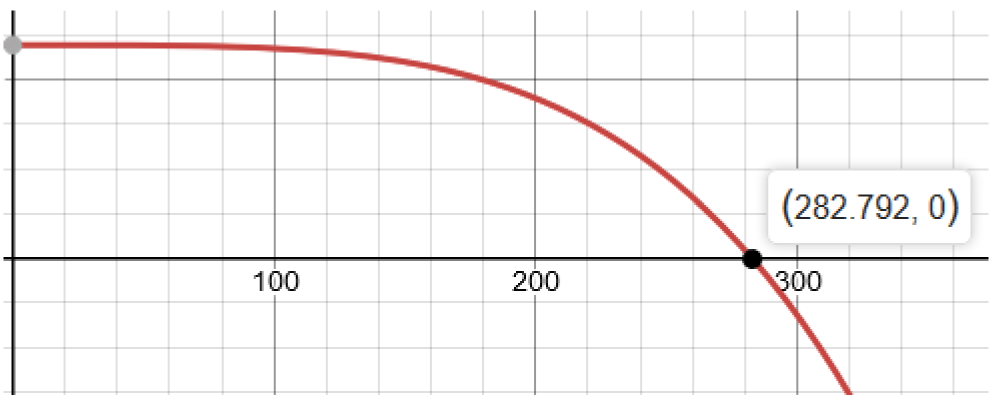
In fact, it is traditional to formulate the equation in terms of energy densities (Wm^{-2}). Recall that E_{in} and E_{out} are energies, so they are expressed in units of watts (W). To convert to energy densities, we need to divide by the Earth's surface area (πR^2). In terms of energy densities, the *temperature evolution equation* is
$C\frac{dT}{dt} = Q(1-\alpha) - \varepsilon\sigma T^4$

This is an *ordinary differential equation* (ODE) for the temperature T as a function of time t. The constant C is the *planetary heat capacity*, which connects the rate of change of the temperature to energy densities.

Question. What is the dimension of C?
[Answer: Joule per kelvin.]

Eq. (5) is an ODE of the type $dT/dt = f(T)$. A visual representation helps us to understand how the Earth's temperature changes when the balance of the incoming and outgoing energy is perturbed.
Sketch the graph of $f(T) = (1 - \alpha)Q - \varepsilon\sigma T^4$
taking $\varepsilon = 0.66$ and $\alpha = 0.3$ and ignoring the constant C since it does not affect the solution of $f(T) = 0$.
Then use the graph to answer the following questions.

Question.



- What does the vertical axis represent in the physical world? [Answer: Rate at which the temperature changes.]
- What is the zero of $f(T)$ in the range between 200 K and 400 K? Where have we seen this value before? What does it represent? [Answer: $f(T) = 0$ for $T = 288$ K. This equilibrium solution of the ODE is the same as the solution found in the previous section. It corresponds to the current state of the climate.]
- If the temperature is 300 K, do you expect the temperature to increase, decrease, or remain the same? Use the graph to help you.
- If the temperature is 250 K, do you expect the temperature to increase, decrease, or remain the same? Use the graph to help you.

Do the same, taking $\varepsilon = 0.5$, and compare your findings in the two cases.

4.1 Analysis

The graph of f is referred to as the *phase line*. It contains all the information about the dynamics of the system. Consider the case $\alpha = 0.3$ and $\varepsilon = 0.66$, where we found an equilibrium at $T^* = 288\,\mathrm{K}$. If the average temperature T is less than T^*, the Earth's surface will warm up; on the other hand, if T is greater than T^*, it will cool down. If T is exactly equal to T^*, it will stay the same. Thus, after any small perturbation, the average temperature tends to be restored to its equilibrium value T^*. In mathematics, we say that T^* corresponds to a *stable* equilibrium.

Question. Is the equilibrium you found for $\varepsilon = 0.5$ stable? [Answer: Yes.]

Conclusion. We can match the current climate state by taking into account the effect of greenhouse gases. Our model indicates that the current climate state is stable