Involutions by Descents/Ascents and Symmetric Integral Matrices

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Alan Hoffman Fest - my hero

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RSK-correspondence

$\mathfrak{S}_n$: the set of all permutations of $\{1, 2, \ldots, n\}$.

$\mathfrak{Y}_n$: the set of **standard Young tableaux** on n cells: a Ferrers diagram containing each of $\{1, 2, \ldots, n\}$ increasing along rows from left to right and along columns from top to bottom. For example. with $n = 8$,

$$\begin{array}{cccc} 1 & 4 & 5 & 7 \\ 2 & 6 & 8 & \\ 3 & & & \end{array} \qquad \text{(of shape } 4, 3, 1\text{)}.$$

The RSK-correspondence (an algorithm) provides a bijection between $\mathfrak{S}_n$ and pairs of standard Young tableaux of the same shape.

$\sigma \xrightarrow{\text{RSK}} (P, Q)$ (P is the insertion tableau and Q is the recording tableau).

Fact: $\sigma \xrightarrow{\text{RSK}} (P, Q) \implies \sigma^{-1} \xrightarrow{\text{RSK}} (Q, P)$

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Thus in the RSK-correspondence, $Q = P$, and hence

$$\sigma \xrightarrow{\text{RSK}} (P, P) \longrightarrow P$$

is a bijection between the set $\mathfrak{I}_n$ of involutions of order n and the set $\mathfrak{Y}_n$ of standard Young tableaux on n cells.

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Generating polynomial of Involutions by descents

descent of a permutation $\pi \in \mathfrak{S}_n$: a position i such that $\pi(i) > \pi(i+1)$. $\mathrm{des}(\pi)$ is the **number of descents** of π.

Generating polynomial of the set of involutions $\mathfrak{I}_n$ by descents:

$$I_n(t) = \sum_{\pi \in \mathfrak{I}_n} t^{\mathrm{des}(\pi)} = \sum_{k=0}^{n-1} I(n,k) t^k,$$

where $I(n,k)$ is the number of involutions of order n with k descents. The first few $I_n(t)$ are (see the on-line encycl. of integer sequences):

$$I_1(t) = 1,$$
$$I_2(t) = 1 + t,$$
$$I_3(t) = 1 + 2t + t^2,$$
$$I_4(t) = 1 + 4t + 4t^2 + t^3,$$
$$I_5(t) = 1 + 6t + 12t^2 + 6t^3 + t^4.$$

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\end{aligned}$$

Known Properties of $I_n(t)$, that is, of the sequence $I(n,0), I(n,1), \ldots, I(n,n-1)$

- symmetric (Strehl, 1981)
- unimodal (Dukes (2006) and Guo & Zeng (2006)
- not log-concave (which would have implied unimodal) (Barnabei, Bonetti & Silimbani (2009)

Descents in Permutation Matrices

Example: Let $n = 5$ and let $\pi = (2, 4, 1, 5, 3)$ with two descents: $4, 1$ and $5, 3$. The corresponding permutation matrix is

$$\begin{bmatrix} & 1 & & & \\ & & & 1 & \\ 1 & & & & \\ & & & & 1 \\ & & 1 & & \end{bmatrix}$$

A descent in terms of the corresponding permutation matrix means that the 1 in some row $i + 1$ is in an earlier column than the 1 in row i.

An involution of order n corresponds to an $n \times n$ symmetric permutation matrix. Thus $I(n, k)$ **counts the number of** $n \times n$ **symmetric permutation matrices with** k **descents**.

Theorem on the Involution Polynomial $I_n(t)$

Recall:

$$I_n(t) = \sum_{k=0}^{n-1} I(n,k)t^k,$$

where $I(n,k)$ is the number of involutions of order n with k descents.

Let $\mathfrak{T}(n,i)$ be the of $i \times i$ **symmetric** matrices with **nonnegative integral entries** with **no zero rows or columns** and **sum of entries equal to n**, and let

$$T(n,i) = |\mathfrak{T}(n,i)|.$$

Theorem: $I_n(t) = \sum_{i=1}^{n} T(n,i)t^{i-1}(1-t)^{n-i}$.

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Making the substitution, $t = \frac{x}{1+x}$, this is equivalent to

$$\sum_{k=0}^{n-1} I(n,k)x^{k+1}(1+x)^{n-1-k} = \sum_{i=1}^{n} T(n,i)x^i,$$

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Equivalent Formulation, solving for $T(n,i)$

$$T(n,i) = \sum_{k=0}^{i-1} I(n,k)\binom{n-1-k}{i-1-k} \quad (i = 1,2,\ldots,n).$$

A permutation with k descents has $(n-1-k)$ **ascents**, and a little manipulation shows that this equation is equivalent to

$$\boxed{T(n,i) = \sum_{j=n-i}^{n-1} I'(n,j)\binom{j}{n-i} \quad (i = 1,2,\ldots,n),}$$

where $I'(n,j) = |\mathcal{I}'(n,j)|$, that is, the cardinality of the set $\mathcal{I}'(n,j)$ of involutions of $\{1,2,\ldots,n\}$ with j ascents.

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The Identity

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where $I'(n,j)$ is the number of $n \times n$ **symmetric permutation matrices with j ascents**, and $T(n,i)$ is the number of $i \times i$ **symmetric matrices with nonnegative integral entries with no zero rows or columns and sum of entries equal to n**.

This suggests that there may be a nice mapping F_{n-i} from the set $\cup_{j \geq n-i}\mathcal{I}'(n,j)$ (all the $n \times n$ symmetric permutation matrices P with $j \geq n-i$ ascents) to subsets of $\mathcal{T}(n,i)$, such that

$$|F_{n-i}(P)| = \binom{j}{n-i}, \text{ and}$$

$$\{F_{n-i}(P) : P \in \cup_{j \geq n-i}\mathcal{I}'(n,j)\} \text{ is a partition of } \mathcal{T}(n,i).$$

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The Mapping $F_{n-i} : \cup_{j\geq n-i}\mathcal{I}'(n,j) \to \mathcal{P}(\mathcal{T}(n,i))$

(i) Let P be an $n \times n$ symmetric perm. matrix with $j \geq n-i$ ascents.

(ii) Let the ascents occur in the pairs of row indices of P given by

$$\{p_1, p_1{+}1\}, \{p_2, p_2{+}1\}, \ldots, \{p_j, p_j{+}1\} \quad (1 \leq p_1 < p_2 < \cdots < p_j < n).$$

(iii) Choose a set X of $(n-i)$ of these pairs.

(iv) The chosen pairs determine an ordered partition $U_1, U_2, \ldots, U_i$ of $[n]$ into maximal sets of consecutive integers: if $U_k = \{m, m+1, m+2, \ldots, q-1, q\}$ (m and q depend on k) , then with $U_k^* = \{\{m, m+1\}, \{m+1, m+2\}, \ldots, \{q-1, q\}\} \subseteq X$ we have $U_1^* \cup U_2^* \cup \cdots \cup U_i^* = X$ and $U_k^* \cap U_l^* = \emptyset$ for $k \neq l$.

(v) The partition in (iv) determines a partition of the rows and columns of P into an $i \times i$ block matrix $[P_{rs}]$. Let $A = [a_{rs}]$ be the $i \times i$ matrix where a_{rs} equals the sum of the entries of P_{rs}.

(vi) Then A is an $i \times i$ symmetric, nonnegative integral matrices without any zero rows and columns, the sum of whose entries equals n.

Since P has $j \geq n-i$ ascents, P gives $\binom{j}{n-i}$ such matrices A.

Example of $F_{n-i} : \cup_{j \geq n-i} \mathcal{I}'(n,j) \to \mathcal{P}(\mathcal{T}(n,i))$

Example: Let $n = 8$ and consider the involution $5, 7, 8, 6, 1, 4, 2, 3$ with corresponding symmetric permutation matrix P. The ascents occur in the pairs of positions (row indices) $\{1,2\}, \{2,3\}$, $\{5,6\}$, and $\{7,8\}$. Choosing the pairs $\{1,2\}$, $\{2,3\}$, and $\{7,8\}$, we obtain the partition $U_1 = \{1,2,3\}, U_2 = \{4\}, U_3 = \{5\}, U_4 = \{6\}, U_5 = \{7,8\}$ of $\{1,2,\ldots,8\}$ and corresponding partition of P given by

$$\left[\begin{array}{ccc||c||c||c||cc}
 & & & & 1 & & & \\ \hline
 & & & & & & 1 & \\ \hline
 & & & & & & & 1 \\ \hline\hline
 & & & & & 1 & & \\ \hline\hline
1 & & & & & & & \\ \hline\hline
 & & & 1 & & & & \\ \hline\hline
 & 1 & & & & & & \\ \hline
 & & 1 & & & & &
\end{array}\right] \to \left[\begin{array}{c|c|c|c|c}
0 & 0 & 1 & 0 & 2 \\ \hline
0 & 0 & 0 & 1 & 0 \\ \hline
1 & 0 & 0 & 0 & 0 \\ \hline
0 & 1 & 0 & 0 & 0 \\ \hline
2 & 0 & 0 & 0 & 0
\end{array}\right].$$

Inverting F_{n-i}

A: an $i \times i$ symmetric, nonnegative integral matrix with no zero rows and columns where the sum of the entries equals n.
r_k: the sum of the entries in row (and column) k of A.
If A is to result from an $n \times n$ symmetric permutation matrix P as described, then, since P has exactly one 1 in each row and column, it must use the partition of the row and column indices of P into the sets:

$$U_1 = \{1, \ldots, r_1\}, U_2 = \{r_1+1, \ldots, r_1+r_2\}, \ldots, U_i = \{r_1+r_2+\cdots+r_{i-1}+1, n\}.$$

There must be a string of $(r_k - 1)$ consecutive ascents corresponding to the positions in each U_k, where there are ascents or descents in the position pairs:

$$(r_1, r_1+1), (r_1+r_2, r_1+r_2+1), \ldots, (r_1+r_2+\cdots+r_{i-1}, r_1+r_2+\cdots+r_{i-1}+1).$$

Then it can be shown that there is exactly one involution (symmetric permutation matrix) with these restrictions.

Inverting F_{n-i}, an Example

Example: $A = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 1 & 3 \\ 0 & 3 & 0 \end{bmatrix}$ Then $n = 11$, and $r_1 = 2, r_2 = 6, r_3 = 3$. We seek an 11×11 symmetric permutation matrix of the form

0	0							0	0	0
0	0							0	0	0
0	0							0	0	0
0	0							0	0	0
0	0							0	0	0

with one ascent in rows 1 and 2, five ascents in rows 3 to 8, and two ascents in rows 9,10,11. It is easy to see that the only possibility is:

Inverting F_{n-i}, an Example ($n = 11$, $r_1 = 2$, $r_2 = 6$, $r_3 = 3$)

$$\left[\begin{array}{cc||cccccc||ccc}
0 & 0 & 1 & & & & & & 0 & 0 & 0 \\
0 & 0 & & 1 & & & & & 0 & 0 & 0 \\
\hline\hline
1 & & & & & & & & & & \\
 & 1 & & & & & & & & & \\
 & & & & 1 & & & & & & \\
 & & & & & & & & 1 & & \\
 & & & & & & & & & 1 & \\
 & & & & & & & & & & 1 \\
\hline\hline
0 & 0 & & & & 1 & & & 0 & 0 & 0 \\
0 & 0 & & & & & 1 & & 0 & 0 & 0 \\
0 & 0 & & & & & & 1 & 0 & 0 & 0
\end{array}\right],$$

equivalently, the involution

$$3, 4; 1, 2, 5, 9, 10, 11; 6, 7, 8.$$

Notice that the pairs of positions which could be either ascents or descents, namely $\{2, 3\}$ and $\{8, 9\}$, are both descents in this case.

Reference

Enumeration of Involutions by Descents and Symmetric Matrices. RAB and Shi-Mei Ma, *European Journal of Combinatorics*, to appear.

Belated Happy 90th Birthday, May 30