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Effects of Change of Scale on Optimality in a Scheduling Model with Priorities and Earliness/Tardiness Penalties

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Abstract—We consider the effect of changes of scale of measurement on the conclusion that a particular solution to a scheduling problem is optimal. The analysis in this paper was motivated by the problem of finding the optimal transportation schedule when there are penalties for both late and early arrivals, and when different items that need to be transported receive different priorities. We note that in this problem, if attention is paid to how certain parameters are measured, then a change of scale of measurement might lead to the anomalous situation where a schedule is optimal if the parameter is measured in one way, but not if the parameter is measured in a different way that seems equally acceptable. This conclusion about the sensitivity of the conclusion that a given solution to a combinatorial optimization problem is optimal is different from the usual type of conclusion in sensitivity analysis, since it holds even though there is no change in the objective function, the constraints, or other input parameters, but only in scales of measurement. We emphasize the need to consider such changes of scale in analysis of scheduling and other combinatorial optimization problems. We also discuss the mathematical problems that arise in two special cases, where all desired arrival times are the same and the simplest case where they are not, namely the case where there are two distinct arrival times but one of them occurs exactly once. While specialized, these two examples illustrate the types of mathematical problems that arise from considerations of the interplay between scale-types and optimization.

Keywords—Scheduling, Meaningfulness, Measurement, Earliness/tardiness penalties, Modeling.

1. INTRODUCTION

Sensitivity analysis has been used extensively in the literature to study the effects of small changes in the objective function on the optimality of a solution to an optimization problem such as a

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX scheduling problem. However, the choice of scale one uses to measure the input (e.g., kilograms versus pounds) could also affect the optimality of a solution. This issue has not been dealt with very often. A key point of this paper is the analysis of the effect on the optimality of a solution to a scheduling problem when an admissible transformation of scale of measurement is performed on scales used to measure the input to the objective function. We wish to show that considerations of scale change need to play a role in the analysis of scheduling problems and other similar problems of combinatorial optimization. A second main point of the paper is to show how challenging mathematical questions arise from these considerations and to analyze one of them in detail.

In this paper, we consider a scheduling problem that was suggested by problems arising at the Air Mobility Command of the United States Air Force. Similar problems are of natural importance for all transportation companies that move packages or people, in the workplace where we schedule jobs, and so on. To be specific about the problem, we shall consider here, suppose that we wish to move a number of items (equipment, people) by vehicles (planes, trucks, trains, pipelines, etc.) from an origin to a destination. We assume that each item has a desired arrival time at the destination, and that we are penalized in some way for missing that time. The penalty can be applied only for a late arrival or, more generally, for both late and early arrivals, perhaps in a different way. Let us assume that we can only take a certain number of items from origin to destination each time that we schedule a trip (say because we have only a limited number of seats on each plane and only a limited number of planes). In principle, trips can have different lengths, though in this paper we shall restrict ourselves to the simple version of the problem in which all trips have the same length. Our goal is to minimize the total penalty.

We also consider an added complication here, namely that the items have different priorities or status or importance. (Transporting fuel may be more important than transporting blankets, transporting a VIP more important than transporting an ordinary person.) If there are different priorities, the penalty for early or late arrival can depend upon the priority.

The introduction of priorities adds a complication if we take into account the way we measure them. Namely, scales of measurement often have certain arbitrary choices (such as of unit or zero point). If we allow admissible transformations of scale, we should ask if the optimal solution to the scheduling problem remains unchanged. Mahadev, Pekeč and Roberts [1] observed, in the context of single machine scheduling and under several models of penalty functions, that it is possible to have the anomalous situation where an optimal schedule under one scale of measurement is no longer an optimal solution after an admissible transformation of scale. We shall note that under some reasonable assumptions, a similar anomaly occurs in our problem, and we seek to discover conditions under which such anomalies can be avoided. We shall see that even in simple models where greedy-type algorithms can be used to obtain optimal solutions, the mathematical analysis involved in proving the nonexistence of such anomalies can be complex (and considerably more complex than that in [1]). We nevertheless obtain two interesting models for objective functions under which such anomalies can be avoided. Similar points are made in the context of general combinatorial optimization problems by Roberts [2,3]. However, we feel that the point needs to be repeated since not many researchers or practitioners seem to have been exposed to it. Moreover, we feel that it leads, in our present context, to interesting mathematical issues.

A problem analogous to the one we are considering arises in the workplace if we have a number of tasks to perform, a number of processors on which to perform them, each task has a desired completion time, and we are penalized in some way for missing that time. We can apply the penalty only to tasks that end after their desired completion time, or also to tasks that end before this time. Tasks are to be scheduled nonpreemptively, i.e., cannot be interrupted once started. We have only a limited number of processors so we can only schedule a certain number of tasks at any given time. In principle, the tasks are allowed to have varying required processing times, though our assumption of fixed trip length is equivalent to the assumption of constant processing times, and tasks can have different priorities. We seek to schedule tasks to processors so that the total penalty is minimized.

This paper is organized as follows. In Section 2, we introduce some formal terminology about scheduling problems and objective functions, and formulate precisely the problem we shall be considering here. In Section 3, we introduce the theory of scale-type in measurement and give some simple examples to show that admissible changes of scale can lead to changes of the optimal solution in scheduling problems of the type in which we are interested here. In Section 4, we make some remarks about the special case of constant desired arrival times or "common due dates". In Section 5, we turn our attention to nonconstant desired arrival times, and we discuss one rather special example of such a problem to illustrate the technical difficulties involved in the analysis of invariance of optimality conclusions. Specifically, we consider the problem where all but one of the arrival times are the same. For this problem, we present conditions sufficient to guarantee that an optimal solution remains optimal after admissible change of scale. The problem considered in this section is quite specialized. The more general problem of arbitrary nonconstant desired arrival times is quite complicated, and it is hard to draw general conclusions about the types of issues with which we are concerned. We feel that analysis of this special case illustrates our points and at the same time shows some interesting mathematical issues that one is led to through considerations of the effect of change of scale on conclusions of optimality. In Section 6, we discuss for a more general penalty function, the special case where the common desired arrival time for all but one of the items is 1. Finally, Section 7 is devoted to a discussion of open problems and future directions for investigation.

2. SCHEDULES AND OBJECTIVE FUNCTIONS

To formulate our problem precisely, let us suppose that there are n items to be transported, and for $i = 1, \ldots, n$, let w_i denote some measure of the priority or status of i. We always assume that priority is a positive real number. Suppose that there is a desired arrival time d_i for the i^{th} item. For simplicity, we assume that the transport arrives only at positive integer times, called **timeslots** (thought of as the first time period, second time period, etc.). Thus, we assume that each d_i is a positive integer. In this paper, the transportation time will be assumed to be the same for each trip. The problem with differing transportation or processing times is of considerable importance and has been widely studied. At any given arrival time, there is a certain available capacity for transportation (number of seats on aircraft, number of processors for processing the task), and for simplicity, we assume that this is a fixed positive integer c. Thus, at any given time, at most c items can arrive at the destination. A **schedule** S is an assignment of a positive integer arrival (completion) time C_i to each item i, subject to the constraint that the number of C_i that can equal a given integer is at most c. We shall let $\mathbf{C}(S) = (C_1, \ldots, C_n) = \mathbf{C}$, $\mathbf{d} = (d_1, \ldots, d_n)$, and $\mathbf{w} = (w_1, \ldots, w_n)$. To compare one schedule to another, we use objective functions

$$F(C_1, \ldots, C_n, w_1, \ldots, w_n, d_1, \ldots, d_n) = F(\mathbf{C}, \mathbf{w}, \mathbf{d})$$

that depend upon the schedule $\mathbf{C}(S)$ and the problem input $\mathbf{w}$ and $\mathbf{d}$, and we seek to minimize these objective functions that are thought of as expressing a penalty for early and (possibly) late arrivals. When the problem input is understood, we denote $F(\mathbf{C}, \mathbf{w}, \mathbf{d})$ by $F(\mathbf{C})$.

In this paper, we shall consider only objective functions that are **summable** in the sense that they can be expressed as $F(\mathbf{C}, \mathbf{w}, \mathbf{d}) = \sum_{i=1}^{n} g(C_i, w_i, d_i)$. We shall also emphasize summable objective functions that are **separable** in the sense that g can be expressed as

$$g(C, w, d) = \begin{cases} h_t(w) f(C, d), & \text{if } C \geq d, \\ h_e(w) f(C, d), & \text{otherwise,} \end{cases}$$

for functions h_t and h_e defined on the positive reals and $f(C, d)$ defined on $\mathbb{N} \times \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers. We shall assume throughout that $h_i(w) > 0$ for $i = t, e$, for all $w > 0$. Often we will assume that the objective function is **symmetric** in the sense that $h_t = h_e$.

To give an example, the objective function

$$F(\mathbf{C}, \mathbf{w}, \mathbf{d}) = \sum_{i=1}^{n} w_i |C_i - d_i|$$

is summable, separable, and symmetric. In this objective function, we simply use the weighted sum of the deviations between desired and scheduled arrival times, weighted by the priority of the items. Here, $f(C,d) = |C-d|$ and $h_t(w) = h_e(w) = w$. A similar objective function arises in single machine scheduling with earliness and tardiness penalties and "noncommon weights" and "symmetric penalties". It is studied, for example, by Cheng [4], Emmons [5], Quaddus [6], Bector, Gupta and Gupta [7], Baker and Scudder [8], Ahmed and Sundararaghavan [9], Hall and Posner [10], and Hoogeveen and van de Velde [11]. A variant of our first example is the objective function in which we use

$$f(C,d) = \begin{cases} |C-d|, & \text{if } C \geq d, \\ 0, & \text{otherwise.} \end{cases}$$

Here, we do not penalize early arrivals. Still a third variant is the simple objective function in which we use

$$f(C,d) = \begin{cases} 1, & \text{if } C \neq d, \\ 0, & \text{otherwise.} \end{cases}$$

Here, we do not penalize long deviations from the desired arrival time more than short ones. Still another simple objective function of interest is the one in which we use

$$f(C,d) = \begin{cases} \gamma|C-d|, & \text{if } C \geq d, \\ \delta|C-d|, & \text{otherwise,} \end{cases}$$

where γ and δ are fixed positive reals. If h_t and h_e are constant functions, this objective function corresponds to the case of "nonsymmetric penalties" with "common weights" in single machine scheduling with earliness and tardiness penalties. It is studied, for example, by Panwalkar, Smith and Seidmann [12], Bagchi, Chang and Sullivan [13], and Emmons [5]. There has been some interest in the literature in nonlinear objective functions. An example would be the simple objective function in which we define $f(C,d) = |C-d|^2$.

The scheduling problem, as we have formulated is close to the scheduling problems that are widely studied in the literature, and we have tried to use notation consistent with that used in the literature. The main difference is how we handle priorities. Some survey papers on scheduling with objective functions are very similar to those we use by Abdul-Razaq, Potts and van Wassenhove [14], Baker and Scudder [15], and Koulamas [16].

3. MEANINGFULNESS OF THE CONCLUSION OF OPTIMALITY

It will be useful to adopt the language of measurement theory in discussing priority measurement. In using scales of measurement, we often make somewhat arbitrary choices such as of unit or zero point. One speaks of **admissible transformations** of scale as transformations of scale values that result from changes in these arbitrary choices. When the admissible transformations of scale correspond exactly to multiplication by a positive constant (as in the measurement of mass, such as when we switch from pounds to kilograms), we say we have a **ratio scale.** When the admissible transformations of scale correspond exactly to multiplication by a positive constant and addition of another constant (not necessarily positive) (as in the measurement of temperature, such as when we switch from fahrenheit to centigrade), we say we have an **interval scale.** In some cases, any strictly increasing transformation of scale is admissible, and in this case, we speak of an **ordinal scale.** The theory of scale-type was introduced into measurement theory by Stevens [17–19]. A general introduction to measurement theory can be found in the books by Krantz *et al.* [20], Roberts [21], Suppes *et al.* [22], and Luce *et al.* [23].

If the truth of a conclusion can depend upon some arbitrary choices involving scales of measurement, as for example about units or zero points, we would probably not want to put much weight behind that conclusion. In measurement theory, we call a statement using scales **meaningful** if its truth or falsity is unchanged after applying admissible transformations to all of the scales in the statement. Mahadev, Pekeč and Roberts [1] show that in the case of the scheduling problem with earliness and tardiness penalties and without priorities, the conclusion that one schedule is optimal can be meaningless. Whether or not it is meaningful depends upon how we measure weights that arise in the objective functions. If priorities are a factor, then meaningfulness should also depend upon how we measure priorities. For further information about meaningful statements, the reader is referred to Roberts [3,24] and Luce *et al.* [23]. For other applications of this concept to problems of combinatorial optimization like the scheduling problems discussed here, see Roberts [2] and Cozzens and Roberts [25]. To give a simple example of the application of measurement theoretical ideas to the scheduling problem we have considered, let us suppose that measurement of priority is on a ratio scale. Then, we can ask whether a change of unit changes the conclusion that a given schedule is optimal. In our notation, this amounts to the question: if $F(\mathbf{C},\mathbf{w},\mathbf{d}) \le F(\mathbf{D},\mathbf{w},\mathbf{d})$ for all schedules $\mathbf{D}$ and α is a positive real number, then is $F(\mathbf{C},\alpha\mathbf{w},\mathbf{d}) \le F(\mathbf{D},\alpha\mathbf{w},\mathbf{d})$ for all schedules $\mathbf{D}$? If F is a summable, separable objective function and if each h_i satisfies

$$h_i(\alpha w) = K(\alpha)h_i(w), \qquad \text{for } \alpha > 0 \text{ and } K(\alpha) > 0, \tag{3.1}$$

then if priority is measured on a ratio scale, it is meaningful to conclude that schedule $\mathbf{C}$ is optimal, i.e., the question has an affirmative answer. This is because $F(\mathbf{D},\alpha\mathbf{w},\mathbf{d}) = K(\alpha)F(\mathbf{D},\mathbf{w},\mathbf{d})$ for all $\mathbf{D}$. (Equation (3.1) arises in measurement theory (in many applied contexts, and originally in psychophysics). See [3,21,26–28] for details.)

By way of contrast, suppose we have the summable, separable, symmetric objective function where $c = 1$, $f(C,d) = |C-d|$, and $h_i(w) = 2^{w-1}$ for all w, $i = 1,2$. If $\mathbf{w} = (1,2,2,2)$ and $\mathbf{d} = (1,2,2,2)$, then the conclusion that schedule $\mathbf{C}$ is optimal is meaningless if priorities are measured on a ratio scale. If $\mathbf{C} = (1,2,3,4)$, then $F(\mathbf{C},\mathbf{w},\mathbf{d}) = 6$ and, it is easy to check, $\mathbf{C}$ is optimal. However, after a change of scale from $\mathbf{w}$ to $\alpha\mathbf{w} = 2\mathbf{w}$, $F(\mathbf{C},2\mathbf{w},\mathbf{d}) = 24$, while $F(\mathbf{D},2\mathbf{w},\mathbf{d}) = 22$, where $\mathbf{D} = (4,1,2,3)$. Thus, $\mathbf{D}$ has a smaller penalty than $\mathbf{C}$. The reader should note that we are not asserting that the objective function we have just analyzed arises in practical scheduling problems. In particular, the priority functions $h_i(w) = 2^{w-1}$ might be unrealistic. However, we are using this example to illustrate the point. It is not hard to show that summable, separable, symmetric objective functions with $f(C,d) = |C-d|$ and linear $h_i(w)$, which are much more realistic, lead to meaningful conclusions of optimality if priorities are measured on a ratio scale. However, later we shall show that such objective functions can lead to meaningless conclusions of optimality if priorities are measured on an interval scale.

It is a reasonable goal to give conditions on a scheduling problem $\mathbf{w},\mathbf{d}$ sufficient for the conclusion of optimality to be meaningful. This turns out to be a difficult question, as we will illustrate in this paper by considering it for a simple case, namely when the vector $\mathbf{d}$ is of the form $(d,d,\ldots,d,k)$.

We will adopt one convention about scale-type in what follows. Specifically, we only consider possible admissible transformations that do not change the range of possible measurements. For instance, if we have an interval scale but scale values must be positive (as in the case of priority measurement), we admit admissible transformations of the form $\varphi(w) = \alpha w + \beta$, $\alpha > 0$, only if $\alpha w + \beta > 0$ for all relevant values w. If we have an ordinal scale and scale values must be positive, we admit all strictly increasing transformations as admissible, so long as they take all relevant values w into positive values $\varphi(w)$. As it turns out, this assumption would not be needed for the proofs if we would allow the functions h_i to be defined for nonpositive arguments. However, the assertions are not interesting without this requirement since priorities are assumed to be positive.

4. A PRELIMINARY: THE CASE OF CONSTANT DESIRED ARRIVAL TIMES

Recall the example from Section 3 which shows that the conclusion that schedule $\mathbf{C}$ is optimal is meaningless if priorities are measured on a ratio scale even if the objective function is summable, separable, and symmetric. However, in the case of constant desired arrival times, i.e., where $\mathbf{d} = (d, d, \dots, d)$, the conclusion is meaningful under some assumptions, as we have also noted in Section 3 and as will be shown in this section. In the scheduling literature, this is called the problem of common due dates. Suppose that the objective function is summable and separable. It will be useful to use the terminology that such an objective function is **w-increasing** if both h_t and h_e are strictly increasing in w, and **$|C-d|$-increasing** if $f(C,d) = A(|C-d|)$ for A, an increasing function. Then it can be seen that if the objective function is symmetric, w-increasing and $|C-d|$-increasing, a greedy algorithm gives an optimal solution. Moreover, any strictly increasing transformation of scale values from w_i to $\varphi(w_i)$ (that is defined in the sense of leaving all scale values positive) gives the same greedy solution. Hence, the conclusion of optimality is meaningful for ratio scales, interval scales, and more generally, for ordinal scales. To make this precise, consider the following greedy algorithm to compute a schedule.

GREEDY ALGORITHM.

(a) Order the items as $i_1, \dots, i_n$ so that $w_{i_1} \geq \cdots \geq w_{i_n}$.
(b) For $j = 1, \dots, n$, do assign to i_j a timeslot closest to d that is still available. (Note that a timeslot remains available until it is assigned to c items.)

Let us say that $\mathbf{C}$ is a **(d,n)-greedy** solution to the scheduling problem $\mathbf{w}, \mathbf{d}$ if it can be obtained by the above greedy algorithm. For example, if $c = 1$, $d = 3$, and $n = 8$ with $w_1 \geq \cdots \geq w_n$, three examples of (d,n)-greedy solutions are $\mathbf{C} = 3,4,2,5,1,6,7,8$; $\mathbf{D} = 3,4,2,1,5,6,7,8$; $\mathbf{E} = 3,2,4,1,5,6,7,8$. Notice that the sequence $|C_{i_j} - d|$, $j = 1, \dots, n$, takes the values $0,1,1,2,2,\dots,d-1,d-1,d,d+1,d+2,\dots$, when $c = 1$.

Suppose that $\mathbf{C}$ is a schedule on n items. For $1 \leq i$, $j \leq n$, define a new schedule $\mathbf{D} =$ switch $(\mathbf{C};\, i,j)$ by taking

$$D_k = \begin{cases} C_i, & \text{if } k = j, \\ C_j, & \text{if } k = i, \\ C_k, & \text{otherwise.} \end{cases}$$

Similarly, for $1 \leq i \leq n$, and a any positive integer (representing a timeslot) define a new schedule $\mathbf{D} =$ move $(\mathbf{C};\, i, a)$ by taking

$$D_k = \begin{cases} a, & \text{if } k = i, \\ C_k, & \text{otherwise.} \end{cases}$$

Notice that if c is the capacity, then in order for $\mathbf{D} =$ move $(\mathbf{C};\, i,a)$ to be a valid schedule, the timeslot a should have been used at most $c-1$ times by $\mathbf{C}$. On the other hand, $\mathbf{D} =$ switch $(\mathbf{C};\, i,j)$ remains valid whenever $\mathbf{C}$ is. In what follows, we will usually assume that the objective function is summable, separable, and symmetric. In this case, we use $h(w)$ in place of $h_t(w)$ and $h_e(w)$.

LEMMA 4.1. *Let F be a summable, separable, symmetric objective function with $f(C,d) = A(|C-d|)$ for some function A. Then, we have*

(a) $F(\mathbf{C}) - F(\mathbf{D}) = h(w_i)[A(|C_i - d_i|) - A(|C_j - d_i|)] + h(w_j)[A(|C_j - d_j|) - A(|C_i - d_j|)]$ *if* $\mathbf{D} =$ switch $(\mathbf{C};\, i,j)$; *and*
(b) $F(\mathbf{C}) - F(\mathbf{D}) = h(w_i)[A(|C_i - d_i|) - A(|a - d_i|)]$ *if* $\mathbf{D} =$ move $(\mathbf{C};\, i,a)$.

PROOF. Straightforward. ∎

If $\mathbf{C}$ and $\mathbf{D}$ are schedules, let us say that $\mathbf{C} \equiv \mathbf{D}$ if $\mathbf{D} =$ switch $(\mathbf{C}; i, j)$ for i, j such that $d_i = d_j$ and $w_i = w_j$. Let us say that $\mathbf{C} \sim \mathbf{D}$ if $\mathbf{C} \equiv \mathbf{D}$ or there are schedules $\mathbf{C}_1, \ldots, \mathbf{C}_m$ so that $\mathbf{C} \equiv \mathbf{C}_1$, $\mathbf{C}_1 \equiv \mathbf{C}_2, \ldots, \mathbf{C}_m \equiv \mathbf{D}$. Note that if $\mathbf{C} \sim \mathbf{D}$ with respect to problem $\mathbf{w}$, $\mathbf{d}$, then $\mathbf{C} \sim \mathbf{D}$ with respect to problem $\varphi(\mathbf{w}), \mathbf{d}$ for any function φ.

It is clear that

(a) $\sim$ is an equivalence relation,
(b) $\mathbf{C} \sim \mathbf{D}$ implies that $\mathbf{C}$ and $\mathbf{D}$ use the same set of arrival times,
(c) if $\mathbf{C}$ and $\mathbf{D}$ are greedy solutions of the same constant desired arrival time problem then $\mathbf{C} \sim \mathbf{D}$, and
(d) if φ is any transformation of priority scale, then $\mathbf{C} \sim \mathbf{D}$ implies that $F(\mathbf{C}, \varphi(\mathbf{w}), \mathbf{d}) = F(\mathbf{D}, \varphi(\mathbf{w}), \mathbf{d})$ for any summable F.

Let $\mathbf{C}$ be a schedule for any scheduling problem. We say that (i, j) is a **reversing pair** for $\mathbf{C}$ if $w_i > w_j$ and $|C_i - d_i| > |C_j - d_j|$.

LEMMA 4.2. *Let $\mathbf{C}$ be a schedule for the constant desired arrival time problem, and suppose F is summable, separable, symmetric, w-increasing, and $|C - d|$-increasing. Then if $\mathbf{C}$ is optimal, it has no reversing pairs.*

PROOF. This is straightforward from Lemma 4.1a since all d_i are the same. ∎

THEOREM 4.3. *Suppose that the objective function is summable, separable, symmetric, w-increasing, and $|C - d|$-increasing. Then $\mathbf{C}$ is an optimal solution for the constant desired arrival time problem if and only if $\mathbf{C}$ is a (d, n)-greedy solution.*

PROOF. Suppose $\mathbf{C}$ is optimal. Then by Lemma 4.2, $\mathbf{C}$ has no reversing pairs. Order the items as $i_1, \ldots, i_n$ so that

$$|C_{i_1} - d| \leq |C_{i_2} - d| \leq \cdots \leq |C_{i_n} - d| \quad \text{and} \quad w_{i_1} \geq \cdots \geq w_{i_n}.$$

This is possible since there are no reversing pairs.

Let $\mathbf{D}$ be a (d, n)-greedy solution obtained using this ordering of the items. Then $|C_i - d| \geq |D_i - d|$ for $i = 1, \ldots, n$ and

$$F(\mathbf{C}, \mathbf{w}, \mathbf{d}) = \sum_{i=1}^{n} h(w_i) A(|C_i - d|) \geq \sum_{i=1}^{n} h(w_i) A(|D_i - d|) = F(\mathbf{D}, \mathbf{w}, \mathbf{d}),$$

with equality if and only if $|C_i - d| = |D_i - d|$ for $i = 1, 2, \ldots, n$. Since $\mathbf{C}$ is optimal, we have equality for all i and so, $\mathbf{C}$ is (d, n)-greedy.

Conversely suppose $\mathbf{C}$ is (d, n)-greedy, and $\mathbf{D}$ is optimal. Then by the above reasoning, $\mathbf{D}$ is (d, n)-greedy, and hence, $\mathbf{C} \sim \mathbf{D}$. Therefore, $\mathbf{C}$ is also optimal. ∎

COROLLARY 4.4. *Suppose that the objective function is summable, separable, symmetric, w-increasing, and $|C - d|$-increasing. Then the statement that $\mathbf{C}$ is an optimal solution for the constant desired arrival time problem is meaningful if priorities are measured on an ordinal scale.*

PROOF. If $\varphi(w)$ is strictly increasing, then the ordering of the items is preserved in the greedy algorithm. Thus, $\mathbf{C}$ is (d, n)-greedy for $\mathbf{w}, \mathbf{d}$ if and only if it is (d, n)-greedy for $\varphi(\mathbf{w}), \mathbf{d}$, and Theorem 4.3 applies. ∎

Of course, in the context of Corollary 4.4, the same conclusion is meaningful for interval scales and ratio scales since it is true for ordinal scales. The reader should note that in this proof, as in subsequent proofs, we do not explicitly use the convention that we only consider φ's such that $\varphi(w) > 0$ for all relevant values of w. However, the problem $\varphi(\mathbf{w}), \mathbf{d}$ is not defined if this is not the case since priorities are supposed to be positive. Moreover, the objective function is also not defined since h_i is only defined on positive priorities.

5. THE $(d, d, \dots, d, k)$ PROBLEM

A similar meaningfulness problem arises if we have nonconstant **d**. It turns out that sometimes nonconstant **d** gives rise to meaningful conclusions of optimality under interval scale, even ordinal scale priority measurement for a variety of objective functions. We shall devote most of the rest of this paper to the analysis of one such **d**, namely $\mathbf{d} = (d, d, \dots, d, k)$, where $k \neq d$ and there are n components. We shall assume without loss of generality that $w_1 \geq \cdots \geq w_{n-1}$. Under these hypotheses, we call the scheduling problem a **(d, k)-problem**. A (d, k)-problem is perhaps the simplest nonconstant **d** problem and it serves to illustrate how challenging the mathematical issues become.

In the case of ratio scale priority measurement, if we use the objective function F defined by

$$(\mathbf{C}, \mathbf{w}, \mathbf{d}) = \sum_{i=1}^{n} w_i |C_i - d_i|, \tag{5.1}$$

the conclusion of optimality is meaningful for a (d, k)-problem. This follows from our earlier observation that summability, separability, and equation (3.1) imply meaningfulness. However, the conclusion of optimality can be meaningless with other objective functions, as we have observed in Section 3. Even with the objective function of equation (5.1), our conclusion under ratio scales does not follow for interval scales. To see why, consider the following example.

EXAMPLE 1. $c = 1$, $\mathbf{w} = (9, 9, 9, 1)$, and $\mathbf{d} = (2, 2, 2, 1)$. Then it is easy to see that the schedule $\mathbf{C} = (2, 1, 3, 4)$ is optimal. However, after the admissible transformation $\varphi(w) = \alpha w + \beta$ given by $\alpha = 1/8$, $\beta = 7/8$, we have $\alpha\mathbf{w} + \beta = (2, 2, 2, 1)$ and $F(\mathbf{C}, \alpha\mathbf{w} + \beta, \mathbf{d}) = 7$, while $F(\mathbf{D}, \alpha\mathbf{w} + \beta, \mathbf{d}) = 6$ for $\mathbf{D} = (2, 3, 4, 1)$.

We shall show that under certain hypotheses, the conclusion of optimality for a (d, k)-problem is meaningful for interval scales and even for ordinal scales.

For a (d, k)-problem, it is useful to study the class $S(x)$ of all schedules $\mathbf{C}$ for which $C_n = x$. It is clear that if $\mathbf{C}$ is optimal for a given scheduling problem and $C_n = x$, then $\mathbf{C}$ is optimal among schedules in $S(x)$. Thus, it suffices to look for the optimal schedule among those schedules that are optimal in $S(x)$ for some x.

Fixing x, we call a solution $\mathbf{C}$ to a (d, k)-problem **x-greedy** if $\mathbf{C}$ is obtained as follows. Let $\mathbf{D}$ be a $(d, n-1)$-greedy solution for the first $n-1$ items using the order $w_1 \geq \cdots \geq w_{n-1}$. Let D_n be a timeslot closest to d that is still available. (There are at most two choices for D_n.) If x appears in the sequence $D_1, \dots, D_{n-1}$, then let $j < n$ be any index such that $D_j = x$. Otherwise, let $j = n$. Then $\mathbf{C}$ is defined by

$$C_i = \begin{cases} D_i, & \text{for } 1 \leq i < j, \\ D_{i+1}, & \text{for } j \leq i < n, \\ x, & \text{for } i = n. \end{cases}$$

REMARK 5.1. Let $|D_n - d| = j$. If $c(2d - 1) > n - 1$, equivalently if $d \geq \lceil (n + c)/2c \rceil$, then schedule $\mathbf{D}$ assigns exactly c items to timeslot d and exactly $2c$ items to timeslots at each of the distances $1, 2, \dots, j - 1$ from d, and the remaining items to timeslots at distance j from d. If $c(2d-1) \leq n-1$, equivalently if $d < \lceil (n + c)/2c \rceil$, then $\mathbf{D}$ assigns c items to each of the timeslots $1, \dots, \lfloor n/c \rfloor$ and the remaining items, if any, to the next timeslot.

All future references to $(d, n - 1)$-greedy solutions $\mathbf{D}$ in this section assume that the solution is obtained using the order $w_1 \geq \cdots \geq w_{n-1}$.

LEMMA 5.2. *For a (d, k)-problem, suppose that F is summable, separable, symmetric, w-increasing, and $|C - d|$-increasing. If $\mathbf{C}$ is optimal in $S(x)$, then $\mathbf{C}$ has no reversing pairs (i, j) with $i, j < n$.*

PROOF. Straightforward. ∎

LEMMA 5.3. *Let* $\mathbf{w}$, $\mathbf{d}$ *be a* (d,k)*-problem, let* F *be a summable, separable, symmetric,* w*-increasing, and* $|C-d|$*-increasing objective function. Then* $\mathbf{C}$ *is optimal for* $S(x)$ *if and only if* $\mathbf{C} \sim \mathbf{D}$ *where* $\mathbf{D}$ *is an* x*-greedy schedule for this problem.*

PROOF. Similar to the proof of Theorem 4.3 applied to the first $n-1$ items. ∎

LEMMA 5.4. *For a given* (d,k)*-problem, let* F *be a summable, separable, symmetric,* w*-increasing, and* $|C-d|$*-increasing objective function. Then the condition that* $\mathbf{C}$ *is optimal for* $S(x)$ *is meaningful if priorities are measured on an ordinal scale.*

PROOF. This follows from Lemma 5.3; the proof is similar to the proof of Corollary 4.4. ∎

Given a (d,k)-problem, let $\mathbf{D}$ be a $(d,n-1)$-greedy solution and $\mathbf{C}$ the corresponding x-greedy solution. If $i<n$ and $|D_i-d|<|C_i-d|$, we say that i is a **pushed item**. We define $PI_{\mathbf{D}}(x)$ to be the set of all pushed items. For example, suppose $c=1$, $\mathbf{d}=(3,3,3,3,3,4)$, $\mathbf{D}=(3,2,4,1,5)$, and $x=4$. Then an x-greedy solution $\mathbf{C}$ is (3,2,1,5,6,4) and $PI_{\mathbf{D}}(x)=\{3,5\}$. Similarly, if $c=2$, $d=3$, and $n=14$, then a $(d,n-1)$-greedy solution is given by $\mathbf{D}=(3,3,2,4,2,4,5,1,1,5,6,6,7)$, and with $x=4$, $PI_{\mathbf{D}}(x)=\{6,10,12\}$.

The following Lemma summarizes some properties of $PI_{\mathbf{D}}(x)$ that we need.

LEMMA 5.5. *Given a* (d,k)*-problem, let* $\mathbf{D}$ *be a* $(d,n-1)$*-greedy solution and* $\mathbf{C}$ *a corresponding* x*-greedy solution. Then,*

(a) *the set* $PI_{\mathbf{D}}(x)$ *is independent of* $\mathbf{D}$ *(we denote this set by* $PI(x)$*);*
(b) *for timeslots* x_1, x_2 *such that* $|x_2-d|\le|x_1-d|$*, we have* $PI(x_1)\subseteq PI(x_2)$*;*
(c) $|(|PI(x+1)|-|PI(x)|)|\le 1$*;*
(d) *if* $d\ge\lceil(n+c)/2c\rceil$*, then* $PI(x)=\emptyset$ *if and only if* $|x-d|\ge\lceil(n-c)/2c\rceil$*. If* $d<\lceil(n+c)/2c\rceil$*, then* $PI(x)=\emptyset$ *if and only if* $x\ge\lceil n/c\rceil$*.*

PROOF. Recall that D_n is a closest timeslot from d that is available after the first $n-1$ items are assigned by a $(d,n-1)$-greedy schedule $\mathbf{D}$ and the timeslots used by schedule $\mathbf{D}$ are as described in Remark 5.1. Observe that for any i such that $|x-d|\le i<|D_n-d|$, the item with largest index among the items at distance i from d is pushed; no other items are pushed. Since all the $(d,n-1)$-greedy solutions use the same order, (a) follows. The rest of the proof is also straightforward from these observations. ∎

Given a (d,k)-problem, let $\mathbf{w}^*=(w_1,\dots,w_{n-1})$ and $\mathbf{d}^*$ be the $(n-1)$-tuple of d's. Suppose F is a summable, separable, symmetric, w-increasing, and $|C-d|$-increasing objective function. If $\mathbf{D}$ is a $(d,n-1)$-greedy solution and $\mathbf{C}$ is the corresponding x-greedy solution, then

$$\begin{aligned} F(\mathbf{C},\mathbf{w},\mathbf{d}) &= F(\mathbf{D},\mathbf{w}^*,\mathbf{d}^*) \\ &\quad+\sum_{i\in PI(x)} h(w_i)\left[A(|C_i-d|)-A(|C_i-d|-1)\right]+h(w_n)A(|x-k|) \\ &= L(\mathbf{w},d,n)+M(\mathbf{w},d,k,n)(x), \end{aligned} \tag{5.2}$$

where $L(\mathbf{w},d,n)=F(\mathbf{D},\mathbf{w}^*,\mathbf{d}^*)$ and

$$M(\mathbf{w},d,k,n)(x)=\sum_{i\in PI(x)} h(w_i)\left[A(|C_i-d|)-A(|C_i-d|-1)\right]+h(w_n)A(|x-k|). \tag{5.3}$$

LEMMA 5.6. *Suppose that the objective function is summable, separable, symmetric,* w*-increasing, and* $|C-d|$*-increasing. Then* $\mathbf{C}$ *is optimal for the* (d,k)*-problem* $\mathbf{w},\mathbf{d}$ *if and only if* $\mathbf{C}$ *is optimal for* $S(C_n)$ *and* $x=C_n$ *minimizes* $M(\mathbf{w},d,k,n)(x)$*.*

PROOF. Certainly if $\mathbf{C}$ is optimal, then it is optimal for $S(C_n)$. So we may assume that $\mathbf{C}$ is optimal for $S(C_n)$. By Lemma 5.3, $\mathbf{C}\sim\mathbf{E}$ where $\mathbf{E}$ is x-greedy for $x=C_n$. Then

$$F(\mathbf{C},\mathbf{w},\mathbf{d})=F(\mathbf{E},\mathbf{w},\mathbf{d})=L(\mathbf{w},d,n)+M(\mathbf{w},d,k,n)(x).$$

But for fixed d, n, $\mathbf{w}$, we have that $L(\mathbf{w}, d, n)$ is independent of x for any x-greedy solution. Thus, $F(\mathbf{C}, \mathbf{w}, \mathbf{d})$ is minimized if and only if $M(\mathbf{w}, d, k, n)(x)$ is minimized. ∎

To state the next theorem, let us say that a (d, k)-problem is **amenable** if $PI(k) = \emptyset$. Necessary and sufficient conditions for amenability are obtained by putting $x = k$ in Lemma 5.5(d).

THEOREM 5.7. *Suppose that the objective function is summable, separable, symmetric, w-increasing, and $|C - d|$-increasing. If a (d, k)-problem is amenable, then the conclusion that $\mathbf{C}$ is optimal for a (d, k)-problem $\mathbf{w}, \mathbf{d}$ is meaningful if priorities are measured on an ordinal scale.*

PROOF. Let φ be strictly increasing. Since A is increasing and $h > 0$,

$$M(\varphi(\mathbf{w}), d, k, n)(x) \geq h(\varphi(w_n)) A(|x - k|) \geq h(\varphi(w_n)) A(0),$$

with both inequalities being equalities if and only if $PI(x) = \emptyset$ and $x = k$. But $PI(k) = \emptyset$ because a (d, k)-problem is amenable, so $x = k$ does achieve both equalities, and no other x achieves the second equality. Thus, by Lemma 5.6, $\mathbf{C}$ is optimal for $\varphi(\mathbf{w}), \mathbf{d}$ if and only if $C_n = k$, and $\mathbf{C}$ is optimal for $S(k)$. ∎

If a (d, k)-problem is not amenable, then we cannot in general conclude meaningfulness under an ordinal scale, even under an interval scale. (See Example 1.) We now turn to situations where we can conclude meaningfulness for ordinal scales even if a (d, k)-problem is not amenable.

LEMMA 5.8. *Consider a (d, k)-problem $\mathbf{w}, \mathbf{d}$ with either*

(i) *$k > d$ or*

(ii) *$k < d$ and $d \geq \lceil (n + c)/2c \rceil$.*

Suppose that the objective function is summable, separable, symmetric, w-increasing, and $|C - d|$-increasing. Suppose $\mathbf{C}$ is an optimal solution and let $x = C_n$. Then if $k > d$, we have $x \geq k$ and if $k < d$, we have $1 \leq x \leq k$.

PROOF. Assume first that $k > d$, but $x < k$. Let $j = |k - x| - 1$. If there is an item $i < n$ such that $C_i = k + j$, then let $\mathbf{D} =$ switch $(\mathbf{C}, i, n)$; otherwise, let $\mathbf{D} =$ move $(\mathbf{C}; n, k + j)$. Then $|C_n - k| = |D_n - k| + 1$ and $C_i = D_i$ or $|C_i - d| > |D_i - d|$. Hence, by Lemma 4.1, $F(\mathbf{C}) > F(\mathbf{D})$, a contradiction.

Now suppose that $k < d$, but $x > k$. Let $j = |k - x| - 1$. By the condition on d and by Remark 5.1, timeslot 1 or $2d - 1$ is available after the first $n - 1$ items are scheduled. Clearly, $\mathbf{C}$ is equivalent in terms of penalty to a schedule $\mathbf{C}'$ for which timeslot 1 is available. Let $j' = \max(1, k - j)$. If there is an item $i < n$ such that $C_i = k - j$, then let $\mathbf{D} =$ switch $(\mathbf{C}; i, n)$; otherwise, let $\mathbf{D} =$ move $(\mathbf{C}'; n, j')$. As in the first part of the proof, it is straightforward to verify, using Lemma 4.1, that $F(\mathbf{C}) > F(\mathbf{D})$, a contradiction. ∎

THEOREM 5.9. *Consider a (d, k)-problem $\mathbf{w}, \mathbf{d}$ with either*

(i) *$k > d$ or*

(ii) *$k < d$ and $d \geq \lceil (n + c)/2c \rceil$.*

Suppose that the objective function is summable, separable, symmetric, and w-increasing, and $f(C, d) = K|C - d| + L$ for $K > 0$. Then the statement that $\mathbf{C}$ is optimal for this problem is meaningful if priorities are measured on an ordinal scale.

PROOF. Note that we may assume that $K = 1$ and $L = 0$. To see why, observe that for any strictly increasing φ,

$$\begin{aligned} F(\mathbf{C}, \varphi(\mathbf{w}), \mathbf{d}) &= \sum_{i=1}^{n} h(\varphi(w_i)) [K|C_i - d_i| + L] \\ &= K \sum_{i=1}^{n} h(\varphi(w_i)) |C_i - d_i| + L \sum_{i=1}^{n} h(\varphi(w_i)), \end{aligned}$$

and so $\mathbf{C}$ minimizes $F(\mathbf{C}, \varphi(\mathbf{w}), \mathbf{d})$ if and only if $\mathbf{C}$ minimizes $\sum_{i=1}^{n} h(\varphi(w_i))|C_i - d_i|$.

By Lemma 5.6, if φ is strictly increasing, $\mathbf{C}$ is optimal for $\varphi(\mathbf{w})$, $\mathbf{d}$ if and only if $\mathbf{C}$ is optimal for $S(C_n)$, and $x = C_n$ minimizes $M(\varphi(\mathbf{w}), d, k, n)(x)$. Thus, it suffices to show that x minimizes $M(\varphi(\mathbf{w}), d, k, n)(x)$ regardless of the choice of strictly increasing φ. Since $K = 1$ and $L = 0$, the function A in the definition of M in equation (5.3) is the identity. Hence, x minimizes $M(\varphi(\mathbf{w}), d, k, n)(x)$ if and only if x minimizes

$$h(\varphi(w_n))\,|x-k| + \sum_{i \in PI(x)} h(\varphi(w_i)) = H(\varphi(\mathbf{w}), n)(x). \tag{5.4}$$

Note that by the convention stated at the end of Section 3, we only consider φ such that $\varphi(w) > 0$ for all w that arise.

CASE 1. $k > d$. By Lemma 5.8, we can assume that $x \geq k$. Let $y = x + 1$. Then by Lemma 5.5, $PI(x) = PI(y)$ or there exists $j(x)$ such that $PI(x) = PI(y) \cup \{j(x)\}$. Let $q(x)$ be $w_{j(x)}$ if $PI(x) \neq PI(y)$ and any number less than w_n otherwise. By equation (5.4)

$$H(\varphi(\mathbf{w}), n)(y) - H(\varphi(\mathbf{w}), n)(x) = \begin{cases} h(\varphi(w_n)) - h\left(\varphi(w_{j(x)})\right), & \text{if } PI(x) \neq PI(y), \\ h(\varphi(w_n)), & \text{if } PI(x) = PI(y). \end{cases} \tag{5.5}$$

Thus, timeslot x is a better choice for C_n than $x+1$ is (i.e., $H(\varphi(\mathbf{w}), n)(y) > H(\varphi(\mathbf{w}), n)(x)$) if and only if $w_n > q(x)$ (since $h \circ \varphi$ is strictly increasing). Note that for x sufficiently large, $PI(x) = PI(y)$ and so $q(x) < w_n$. Hence, $H(\varphi(\mathbf{w}), n)(x)$ is minimized for the smallest $x \geq k$ for which $w_n > q(x)$ and for all $x \geq k$ with $w_n = q(x)$. Clearly, such x are independent of φ.

CASE 2. $k < d$. Again by Lemma 5.8, we can assume that $1 \leq x \leq k$. Note that equation (5.5) still holds for $1 < x \leq k$ and $y = x - 1$ with $j(x)$ as in Case 1. Let $q(x)$ be as in Case 1, with $q(1)$ any number less than w_n, by convention. We find that $H(\varphi(\mathbf{w}), n)(x)$ is minimized for the largest $x \leq k$ for which $w_n > q(x)$ and for all $x \leq k$ with $w_n = q(x)$. Clearly, such x are independent of φ. ∎

REMARK 5.10. If $k < d$ and $d < \lceil (n+c)/2c \rceil$, the conclusion of Theorem 5.9 does not necessarily follow. This follows from Example 1.

REMARK 5.11. The assumption in Theorem 5.9 that f is linear, i.e., $f(C, d) = K|C - d| + L$, cannot be replaced by an assumption that f is quadratic. Consider the summable, separable, symmetric objective function F with $h(w) = w$ and $f(C, d) = |C - d|^2$. Let $c = 1$, $\mathbf{d} = (1, 1, 2)$, $\mathbf{w} = (5, 1, 4)$. Then for $\mathbf{C} = (1, 3, 2)$, $F(\mathbf{C}, \mathbf{w}, \mathbf{d}) = 4$ and it can be shown that $F(\mathbf{E}, \mathbf{w}, \mathbf{d}) > 4$ for any other schedule $\mathbf{E}$. However, for $\varphi(\mathbf{w}) = \mathbf{w} + \mathbf{1} = (6, 2, 5)$, $F(\mathbf{C}, \varphi(\mathbf{w}), \mathbf{d}) = 8$, while $F(\mathbf{D}, \varphi(\mathbf{w}), \mathbf{d}) = 7$ for $\mathbf{D} = (1, 2, 3)$.

REMARK 5.12. We note that Theorems 5.7 and 5.9 do not generalize to the case where $\mathbf{d} = (d, d, \dots, d, k, k)$, even for interval scales. Consider the objective function F defined in equation (5.1), let $c = 1$, $\mathbf{d} = (3, 3, 3, 4, 4)$, $\mathbf{w} = (10, 10, 3, 1, 1)$. Then it is straightforward to show that $\mathbf{C} = (3, 2, 4, 5, 6)$ is an optimal solution. However, if $\varphi(\mathbf{w}) = \mathbf{w} + \mathbf{2} = (12, 12, 5, 3, 3)$, then $F(\mathbf{C}, \varphi(\mathbf{w}), \mathbf{d}) = 26$, while $F(\mathbf{D}, \varphi(\mathbf{w}), \mathbf{d}) = 25$ for $\mathbf{D} = (3, 2, 1, 4, 5)$.

Recall that for summable, separable objective functions, equation (3.1) leads to meaningfulness of the conclusion of optimality in the case of ratio scale priority measurement. It turns out that an equation like it can also be used as an assumption that leads to meaningfulness in the case of ordinal scale priority measurement. Let us say that a summable, separable objective function F is **i-semilinear** for $i = t$ or e if for every $\alpha > 0$ and β, if $w > 0$ and $\alpha w + \beta > 0$, then

$$h_i(\alpha w + \beta) = K_i(\alpha, \beta) h_i(w) + L_i(\alpha, \beta), \qquad K_i(\alpha, \beta) > 0. \tag{5.6}$$

We say that F is **pair semilinear** if it is i-semilinear for $i = t, e$, and K_i is the same function for $i = t, e$, and L_i is the same function for $i = t, e$. (As in the case of equation (3.1), equation (5.6) arises in measurement theory. See [3,21,26–28] for details.) We shall need the following result.

THEOREM 5.13. *(See [27].) Suppose that $h_i : \mathbb{R}^+ \to \mathbb{R}^+$ is a given function for $i = t, e$ and with it F is i-semilinear. Then $h_i(w) = a_i w + b_i$, where $a_i \geq 0$, $b_i \geq 0$ and either $a_i > 0$ or $b_i > 0$. If F is pair semilinear, then $h_t = h_e$.*

By Theorem 5.13, under pair semilinearity, $h_t(w) = h_e(w) = Aw + B$. In the case where $A > 0$, the objective function is symmetric and w-increasing. Thus, these hypotheses of Theorem 5.9 do not need to be assumed. In the case where $A = 0$, h_t and h_e are constants, and so for all $\mathbf{C}$ and all transformations φ, $F(\mathbf{C}, \mathbf{w}, \mathbf{d}) = F(\mathbf{C}, \varphi(\mathbf{w}), \mathbf{d})$. In this case, optimality of $\mathbf{C}$ is clearly preserved under any transformation φ. Thus, we have the following corollary of Theorem 5.9.

THEOREM 5.14. *Consider a (d, k)-problem $\mathbf{w}, \mathbf{d}$ with either*

(i) $k > d$ *or*

(ii) $k < d$ *and* $d \geq \lceil (n + c)/2c \rceil$.

Suppose that the objective function is summable, separable, and pair semilinear, and $f(C, d) = K|C - d| + L$ for $K > 0$. Then the statement that $\mathbf{C}$ is optimal for this problem is meaningful if priorities are measured on an ordinal scale.

REMARK 5.15. It is interesting to note that in Theorem 5.14, the hypothesis of pair semilinearity cannot be replaced by assuming t-semilinearity and e-semilinearity. Suppose that $c = 1$, $\mathbf{d} = (2, 2, 3)$, $\mathbf{w} = (1, 1, 1)$, $h_t(w) = 3 + w$, $h_e(w) = 3w$, $K = 1$, $L = 0$. The schedule $\mathbf{C} = (2, 1, 3)$ is optimal since its penalty is 3 and any schedule has penalty at least 3. However, consider $\varphi(\mathbf{w}) = 7\mathbf{w}$. Then $\mathbf{C}$ has penalty 21 while $\mathbf{D} = (2, 4, 3)$ has penalty 20. The conclusion of the theorem is false, even for ratio scales.

6. THE $(1, 1, \ldots, 1, k)$ PROBLEM

For the special case $d = 1$, we shall observe in this section that we can obtain stronger results than Theorems 5.9 and 5.14. In particular, there is no need to worry about the case $k < d$, since we must have $k > d$. Also, as we shall observe, symmetry need not be assumed, and in fact, $h_e(w)$ can be an arbitrary (positive) function. However, we shall have to be a little stricter about $f(C, d)$. We shall use the terminology **$(1, k)$-problem** for a (d, k)-problem with $d = 1$.

THEOREM 6.1. *Consider a $(1, k)$-problem $\mathbf{w}, \mathbf{d}$ and suppose that the objective function is summable and separable, $h_t(w)$ is strictly increasing in $\mathbf{w}$, and $f(C, d) = K|C - d|$ for $K > 0$. Then the statement that $\mathbf{C}$ is optimal for this problem is meaningful if priorities are measured on an ordinal scale.*

PROOF. Given the objective function F, define a new summable, separable objective function F' by letting $f' = f$, $h'_t = h'_e = h_t$. Then F' is symmetric. Moreover, $k > d$ since $k \neq d$ in a (d, k)-problem, and so the hypotheses of Theorem 5.9 are satisfied. We shall show that $\mathbf{C}$ is optimal for F if and only if $\mathbf{C}$ is optimal for F'. The meaningfulness of the conclusion that $\mathbf{C}$ is optimal for F, then follows from the meaningfulness of the conclusion that $\mathbf{C}$ is optimal for F', which holds by Theorem 5.9.

Let $\mathbf{C}$ be optimal for the $(1, k)$-problem $\mathbf{w}, \mathbf{d}$ under F. We shall show that $C_i \geq d_i$ for all i. Of course, this is trivial for $i < n$. Suppose that $C_n < d_n = k$. If there is an $r < n$ such that $C_r = C_n + 1$, then let $\mathbf{D} =$ switch $(\mathbf{C}; r, n)$; otherwise let $\mathbf{D} =$ move $(\mathbf{C}; n, C_n + 1)$. Then

$$F(\mathbf{C}, \mathbf{w}, \mathbf{d}) - F(\mathbf{D}, \mathbf{w}, \mathbf{d}) = \begin{cases} h_e(w_n)K, & \text{if no } C_r \text{ is } C_n + 1, \\ h_e(w_n)K + h_t(w_r)K, & \text{otherwise,} \end{cases}$$

which is positive since $h_t > 0$, $h_e > 0$, and $K > 0$, contradicting the optimality of $\mathbf{C}$. Hence, we conclude that $C_i \geq d_i$ for all i. It follows that $F'(\mathbf{C}, \mathbf{w}, \mathbf{d}) = F(\mathbf{C}, \mathbf{w}, d)$. Let $\mathbf{D}$ be optimal for F'. Then by Lemma 5.8, since $k > d$, we have $D_n \geq k$. Thus $D_i \geq d_i$ for all i, and hence, $F'(\mathbf{D}, \mathbf{w}, \mathbf{d}) = F(\mathbf{D}, \mathbf{w}, d)$. It follows that $\mathbf{C}$ is optimal for F if and only if $\mathbf{C}$ is optimal for F'. ∎

REMARK 6.2. A similar proof shows that the same conclusion holds if we let $f(C,d) = K|C - d| + L$, $K > 0$, and add the hypothesis that $g(w,k,k) < g(w,k-1,k)$ for all w, or the more general hypothesis that $g(w,C,d) < g(w,D,d)$ if $|C-d| < |D-d|$. This gives us

$$h_e(w_n)f(k-1,k) - h_t(w_n)f(k,k) > 0.$$

REMARK 6.3. The reader should note that the assumption that arrival times must be positive plays a central role in Theorem 6.1. To illustrate this point, modify the example in Remark 5.15 by shifting desired and scheduled arrival times by 1, i.e., taking $\mathbf{d} = (1,1,2)$, $\mathbf{C} = (1,0,2)$, and $\mathbf{D} = (1,3,2)$, while leaving priorities $\mathbf{w} = (1,1,1)$ unchanged. As before, $\mathbf{C}$ is optimal but after change of scale $\varphi(\mathbf{w}) = 7\mathbf{w}$, $\mathbf{D}$ has a smaller penalty than $\mathbf{C}$.

The next result follows from Theorem 6.1 in the same way that Theorem 5.14 follows from Theorem 5.9. We use the observation in Theorem 5.13 that t-semilinearity implies $h_t(w) = Aw + B$, $A \geq 0$.

THEOREM 6.4. *Consider a $(1,k)$-problem $\mathbf{w}, \mathbf{d}$ and suppose that the objective function is summable, separable, and t-semilinear, and $f(C,d) = K|C-d|$ for $K > 0$. Then the statement that $\mathbf{C}$ is optimal for this problem is meaningful if priorities are measured on an ordinal scale.*

7. OPEN QUESTIONS AND FURTHER DIRECTIONS FOR ANALYSIS

In this paper, we have left a number of specific questions for further investigation. Problems with $\mathbf{d} = (d,d,\dots,d,k,k)$ and $\mathbf{d} = (d,d,\dots,d,k_1,\dots,k_m)$ are natural generalizations of the (d,k)-problem. Even a (d,k)-problem when $k < d$ and $d < \lceil (n+c)/2c \rceil$ remains to be analyzed.

As mentioned in Section 2, we assumed that c is constant for simplicity. However, our results easily generalize to the case of nonconstant $c : \mathbb{N} \to \mathbb{N}$ (different aircraft with different numbers of seats might operate on different days; the number of processors available for processing the task might be different due to maintenance schedules). For example, the condition on d in (ii) of Theorem 5.9 becomes

$$\sum_{a=1}^{2d-1} c(a) \geq n.$$

Other problems need to be analyzed for nonconstant c.

Greedy algorithms have played a role in this paper. It should be useful to determine general conditions under which a greedy algorithm gives an optimal solution to the kinds of scheduling problems considered here.

We have assumed throughout that there is a constant transportation time. It would be interesting to analyze problems of the kind we have discussed if transportation times can vary.

In investigating the constant desired arrival time problem and a (d,k)-problem, we looked for situations where the conclusion of optimality was meaningful for interval scales but not ordinal scales, but were unable to find any such examples. It would be interesting to study situations where we obtain interval scale but not ordinal scale meaningfulness.

The special cases of scheduling problems $\mathbf{w}, \mathbf{d}$ investigated in this paper are just some of those which would be interesting to investigate further. It would also be of interest to consider other assumptions about objective functions. More generally, one would like to systematically analyze such questions as the following. Given a scheduling problem and an objective function, for what types of scales of priority is the conclusion of optimality meaningful? Given a scheduling problem and a scale-type for priority measurement, for what types of objective functions is the conclusion of optimality meaningful? Given an objective function and a scale-type for priority measurement, for what types of scheduling problems is the conclusion of optimality meaningful? Many of these questions should lead to challenging mathematical problems.

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